

## Harmonic Oscillator:

Why study harmonic oscillator —

It is not only exactly solvable in classical (as well as quantum theory) but it is a system of great physical relevance. Concepts <sup>are</sup> extended to related problems.

For example, any system having small fluctuation near a stable equilibrium configuration may be described by a single harmonic ~~oscillator~~ <sup>oscillator</sup> or by a collection of decoupled harmonic oscillators.

Example of a single harmonic oscillator is a mass  $m$  coupled to a spring of force constant  $k$ .

For small deformations  $x$ , spring will exert the force given by Hooke's law  $F = -kx$ ,

$k$  is its force constant

and it produce a potential  $U = \frac{1}{2} kx^2$ .

The total energy for this system is  $E = T + V$

$$\text{OR } E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$\text{where } \omega = \sqrt{k/m}.$$

$\omega$  is classical frequency of oscillation.

Mass-spring system is one example of harmonic oscillator. (2)  
Now consider a particle moving in a potential  $U(x)$ .

If particle is placed at one of its minima  $x_0$ , it will remain there in a state of stable static equilibrium.

~~and~~ we can write the dynamics of the particle as it fluctuates by small amounts near  $x=x_0$ .

The Potential  $U(x)$  may be expanded about  $x_0$  by using Taylor series:

$$U(x) = U(x_0) + \left. \frac{dU}{dx} \right|_{x=x_0} (x-x_0) + \frac{1}{2!} \left. \frac{d^2U}{dx^2} \right|_{x=x_0} (x-x_0)^2 + \dots$$

$U(x_0)$  has no physical consequence and may be dropped.

One can choose it as the arbitrary reference point for determining the potential.

$$\text{So } U(x) = \frac{1}{2!} \left. \frac{d^2U}{dx^2} \right|_{x=x_0=0} (x-x_0)^2 + \frac{1}{3!} \left. \frac{d^3U}{dx^3} \right|_{x=x_0=0} (x-x_0)^3 + \dots$$

For small oscillations, we can neglect higher order terms and obtain

$$U(x) = \frac{1}{2} \left. \frac{d^2U}{dx^2} \right|_{x=x_0=0} (x-x_0)^2$$

$$\text{or } U(x) = \frac{kx^2}{2} \frac{d^2U}{dx^2}$$

$$\frac{d^2U}{dx^2} \rightarrow \text{identical to } m\omega^2 = k. \quad \left\{ \text{see the} \right.$$

example on page - 01 }



## Simple Harmonic motion:

A particle is said to perform a simple harmonic motion (SHM) if its acceleration is proportional to its displacement from its equilibrium position and is always or any other point in its path and is always directed towards it.

The force  $F$  for SHM is given by  $F = -kx$

where  $k$  is a positive constant.

Now using Newton's second law of motion, we can write

$$m \frac{d^2x}{dt^2} = -kx$$

$$\text{or } \boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

↑  
differential equation of motion of a simple harmonic motion.

Let  $\frac{k}{m} = \omega^2$ , we can write above equation

$$\frac{d^2x}{dt^2} + \omega^2x = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} = -\omega^2x$$

Multiplying both sides of the equation by  $2 \frac{dx}{dt}$

$$2 \frac{dx}{dt} \frac{d^2x}{dt^2} = -2\omega^2x \frac{dx}{dt}$$

$$\text{or } \int 2 \frac{dx}{dt} \frac{d^2x}{dt^2} dt = - \int 2\omega^2x \frac{dx}{dt} dt \quad (\text{Integrating wrt } t)$$

$$\int \frac{d}{dt} \left( \frac{dx}{dt} \right)^2 dt = - \int \omega^2 \frac{d}{dt} (x^2) dt$$

$$\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + C$$

$\uparrow$   
 constant of integration.

at maximum displacement,  $\frac{dx}{dt} = 0$ , we can write

$$0 = -\omega^2 a^2 + C$$

$$\Rightarrow C = \omega^2 a^2$$

$$\text{Now } \left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + \omega^2 a^2 = \omega^2 (a^2 - x^2)$$

$$\text{or } \frac{dx}{dt} = \omega \sqrt{a^2 - x^2}, \quad \frac{dx}{dt} \rightarrow \text{velocity of the particle at an instant } t.$$

$$\frac{dx}{\sqrt{a^2 - x^2}} = \omega dt$$

integrating above eqn w.r.t.  $t$

$$\sin^{-1} \frac{x}{a} = \omega t + \phi$$

$$\boxed{x = a \sin(\omega t + \phi)} \quad \left\{ \begin{array}{l} \text{displacement of the} \\ \text{particle at an instant } t \end{array} \right\}$$

$a \rightarrow$  amplitude

$\omega t + \phi \rightarrow$  total phase

$\phi \rightarrow$  phase constant

At  $x=0 \rightarrow t=0$ , we have  $\phi=0$ , and can write

$$\boxed{x = a \sin \omega t}$$

When  $x=a$ , at  $t=0$ , we have  $a = a \sin(\phi)$

$$\phi = \pi/2$$

$$\text{or } \boxed{x = a \sin(\omega t + \pi/2)}$$

~~etc~~

$$\text{Time period} - \boxed{T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}} \rightarrow \text{independent of } a, \phi.$$